Question 3

Intent of Question

The primary goals of this question are to assess a student's ability to: (1) recognize the random variable of interest, identify its probability distribution, and calculate a probability for a linear combination of a normal random variable and a constant; (2) use basic probability rules to find a different probability; and (3) use the sampling distribution of the sample mean to find a probability about the mean of three observations.

Solution

Part (a):

Since $M = D + E$ (a normal random variable plus a constant is a normal random variable), we know that M is normally distributed with a mean of 2 feet and a standard deviation of 1.5 feet. Thus,

$$P(M < 0) = P\left(Z < \frac{0-2}{1.5}\right) < P(Z < -1.33) = 0.0918 \text{, where } Z = \frac{M - \mu}{\sigma}.$$

Part (b):

$$P(\text{at least one measurement} < 0) = 1 - P(\text{all three measurements} \geq 0)$$
$$= 1 - (1 - 0.0918)^3$$
$$= 1 - (0.9082)^3$$
$$= 1 - 0.7491$$
$$= 0.2509$$

Part (c):

Let $\bar{X}$ denote the mean of three independent depth measurements taken at a point where the true depth is 2 feet. Since each measurement comes from a normal distribution, the distribution of $\bar{X}$ is normal with a mean of 2 feet and a standard deviation of $\frac{1.5}{\sqrt{3}} = 0.8660$ feet. Thus,

$$P(\bar{X} < 0) = P\left(Z < \frac{0-2}{\frac{1.5}{\sqrt{3}}}\right) < P(Z < -2.31) = 0.0104 \text{, where } Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}.$$

Scoring

Parts (a), (b), and (c) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Part (a) is essentially correct (E) if the student clearly does ALL three of the following:
- identifies the distribution as normal;
- specifies BOTH μ and σ; AND
- calculates the correct probability.

Part (a) is partially correct (P) if the student:
- calculates the correct probability but fails to identify the distribution as normal with BOTH μ and σ specified;
    OR
- correctly identifies the distribution as normal with BOTH μ and σ specified but fails to calculate the correct probability.

Part (a) is incorrect (I) if any of the following occur:
- the student indicates the probability is 0.5 because the random error is symmetric about zero;
    OR
- the student uses a mean of zero and a standard deviation of 1;
    OR
- the student conducts a hypothesis test.

Notes:
- The student may use the distribution of the error, *E*, to solve the problem. That is, finding the area below -2 for a normal distribution with mean 0 and standard deviation 1.5 should be scored essentially correct (E).
    Thus $P(E < -2) = P\left(Z < \dfrac{-2-0}{1.5}\right) < P(Z < -1.33) = 0.0918$, where $Z = \dfrac{E - \mu}{\sigma}$.
- If only the calculator command normalcdf $(-\infty, 0, 2, 1.5)$ is provided along with the probability 0.0912, then the response should be scored as partially correct (P).

Part (b) is essentially correct (E) if the student calculates the correct probability AND:
- correctly applies complement and probability rules using the value obtained in part (a);
    OR
- clearly identifies the distribution as binomial AND specifies BOTH *n* and *p* using the value obtained in part (a).

    Part (b) is partially correct (P) if the student:
- clearly identifies the distribution as binomial AND specifies BOTH *n* and *p*, using the value obtained in part (a), but does not calculate the correct probability;
    OR

- clearly identifies the distribution as binomial AND specifies BOTH n and p using a value of p that is unrelated to the value obtained in part (a) and calculates the correct probability based on their value of p;
      OR
- calculates the correct probability using the value obtained in part (a) but fails to correctly identify the distribution as binomial with BOTH n and p specified;
      OR
- recognizes the solution as the sum of the product of the probabilities of successes and failures, using the answer from part (a), but omits only the binomial coefficients.

Part (b) is incorrect (I) if the student calculates $P(\text{at least one measurement} < 0) = 1 - p^3$, where p is the solution to part (a).

Notes:
- The solution using the binomial distribution with $p = 0.0918$ is:

$$P(\text{at least one measurement} < 0) = P(B = 1) + P(B = 2) + P(B = 3)$$

$$= \binom{3}{1} 0.0918^1 (1 - 0.0918)^2 +$$

$$\binom{3}{2} 0.0918^2 (1 - 0.0918)^1 + \binom{3}{3} 0.0918^3$$

$$= 0.2272 + 0.0230 + 0.0008$$

$$= 0.2510$$

- If only the calculator command $1 - \text{binomcdf}(3, 0.0918, 0)$ is provided along with the probability 0.2509, then the response should be scored as partially correct (P).

Part (c) is essentially correct (E) if the student clearly does ALL three of the following:
- identifies the distribution of the sample mean as normal;
- specifies BOTH $\mu_{\bar{x}}$ and $\sigma_{\bar{x}}$; AND
- calculates the correct probability.

Part (c) is partially correct (P) if the student:
- calculates the correct probability, but fails to identify the distribution of the sample mean as normal with BOTH $\mu_{\bar{x}}$ and $\sigma_{\bar{x}}$ specified;
      OR
- correctly identifies the distribution of the sample mean as normal with BOTH $\mu_{\bar{x}}$ and $\sigma_{\bar{x}}$ specified, but fails to calculate the correct probability.

Part (c) is incorrect (I) if any of the following occur:
- the student uses the same calculation as in part (a);
      OR

- the student uses an incorrect standard deviation (e.g., $\dfrac{1.5}{\sqrt{2}}$ or $\sqrt{3(1.5)}$) ;

    OR
- the student conducts a hypothesis test.

Notes:

- An alternate solution using the sum instead of the mean is:

    Let $\bar{X}$ denote the mean of three independent depth measurements taken at a point where the true depth is 2 feet. Since each measurement comes from a normal distribution, the distribution of the sum of the three measurements, $S = (X_1 + X_2 + X_3)$, is normal with a mean $\mu_S = 6$ feet and a standard deviation

    $\sigma_S = 2.598$ feet $\left(\sigma_S = 3\left(\dfrac{1.5}{\sqrt{3}}\right)\right.$, often calculated as $\left.\sqrt{(1.5)^2 + (1.5)^2 + (1.5)^2}\right)$.

    Thus $P(S < 0) = P\left(Z < \dfrac{0 - 6}{2.598}\right) = P(Z < -2.31) = 0.0104$, where $Z = \dfrac{S - \mu_S}{\sigma_S}$.

- If only the calculator command normalcdf $(-\infty, 0, 2, 0.866)$ is provided along with the probability 0.01046, then the response should be scored as partially correct (P).
- If the student does not consistently specify a correct μ and σ from the same distribution, i.e., for the mean or the sum, the response should be scored at most partially correct (P).

4   Complete Response

All three parts essentially correct

3   Substantial Response

Two parts essentially correct and one part partially correct

2   Developing Response

Two parts essentially correct and no parts partially correct

OR

One part essentially correct and two parts partially correct

OR

Three parts partially correct

1   Minimal Response

One part essentially correct and one part partially correct

OR

One part essentially correct and no parts partially correct

OR

No parts essentially correct and two parts partially correct

3. The depth from the surface of Earth to a refracting layer beneath the surface can be estimated using methods developed by seismologists. One method is based on the time required for vibrations to travel from a distant explosion to a receiving point. The depth measurement (M) is the sum of the true depth (D) and the random measurement error (E). That is, $M = D + E$. The measurement error (E) is assumed to be normally distributed with mean 0 feet and standard deviation 1.5 feet.

(a) If the true depth at a certain point is 2 feet, what is the probability that the depth measurement will be negative?

$M < 0$

$M_m = 2 + 0$
$\sigma_m = 1.5$
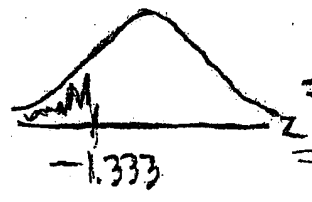
$M = D + E$

$P(M < 0) = P(D + E < 0)$

$= P\left(z < \frac{0-2}{1.5}\right)$

$= P(z < -1.333)$

$= ncdf(-10, -1.333)$

$= \boxed{0.0913}$

-1.333

(b) Suppose three independent depth measurements are taken at the point where the true depth is 2 feet. What is the probability that at least one of these measurements will be negative?

$P(positive) = 1 - .0913 = .9087$

$P(at least one neg) = 1 - P(no neg)$
$= 1 - P(+) \cdot P(+) \cdot P(+)$

$+, +, +$

$= 1 - (.9087)^3$

$= \boxed{0.25}$

(c) What is the probability that the mean of the three independent depth measurements taken at the point where the true depth is 2 feet will be negative?

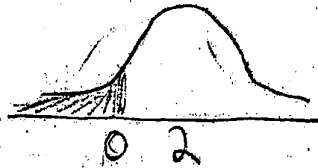$M = \frac{2(3)}{3} = 2$

$\sigma = \frac{\sqrt{3(1.5)^2}}{3} = 0.866$
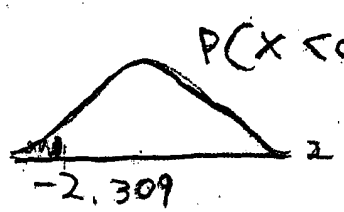
$P(\bar{x} < 0) = P\left(z < \frac{0-2}{.866}\right) = P(z < -2.309)$

$= ncdf(-10, -2.309)$

$= \boxed{0.010}$

-2.309   2

GO ON TO THE NEXT PAGE.

-10-

3. The depth from the surface of Earth to a refracting layer beneath the surface can be estimated using methods developed by seismologists. One method is based on the time required for vibrations to travel from a distant explosion to a receiving point. The depth measurement (M) is the sum of the true depth (D) and the random measurement error (E). That is, $M = D + E$. The measurement error (E) is assumed to be normally distributed with mean 0 feet and standard deviation 1.5 feet.

(a) If the true depth at a certain point is 2 feet, what is the probability that the depth measurement will be negative?

$$ncdf(-1e99, 0, 2, 1.5)$$

P = .09121 that the value will be negative

(b) Suppose three independent depth measurements are taken at the point where the true depth is 2 feet. What is the probability that at least one of these measurements will be negative?
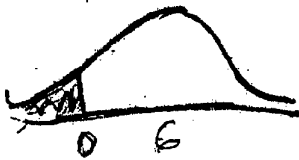
$$_3C_1 (.09121)^1 (1-.09121)^2 = .22599$$

$$_3C_2 (.09121)^2 (1-.0912)^1 = .02269$$

$$_3C_3 (.09121)^3 (1-.0912)^0 = +7.588 \times 10^{-4}$$

$$P = .2494 \text{ at least one will be negative}$$

(c) What is the probability that the mean of the three independent depth measurements taken at the point where the true depth is 2 feet will be negative?

$$\sigma = \sqrt{1.5^2 + 1.5^2 + 1.5^2}$$

$$\sigma = 2.598$$

$$ncdf(-e99, 0, 6, 2.598)$$

P = .01046 that the mean will be negative

-10-

3. The depth from the surface of Earth to a refracting layer beneath the surface can be estimated using methods developed by seismologists. One method is based on the time required for vibrations to travel from a distant explosion to a receiving point. The depth measurement (M) is the sum of the true depth (D) and the random measurement error (E). That is, $M = D + E$. The measurement error (E) is assumed to be normally distributed with mean 0 feet and standard deviation 1.5 feet.

(a) If the true depth at a certain point is 2 feet, what is the probability that the depth measurement will be negative?

$$M = D + E$$
$$M = 2 + E$$
for M to be negative, $E < -2$

$$P(E < -2) = P\left(Z < \frac{-2-0}{1.5}\right) = P(Z < -1.33) = \boxed{0.0912}$$

(b) Suppose three independent depth measurements are taken at the point where the true depth is 2 feet. What is the probability that at least one of these measurements will be negative?

Let X be a binomial var, the number of measurements that are negative.
Conditions for binomial: ① Fixed n # of observations — yes, 3
② P(success) same for all — yes
③ Independence of observations — yes
④ Only 2 outcomes — yes, + or −

$$P(X \geq 1) = 1 - P(X = 0)$$
$$= \boxed{0.249}$$

(c) What is the probability that the mean of the three independent depth measurements taken at the point where the true depth is 2 feet will be negative?

$$P\left(\frac{1}{3}(M_1 + M_2 + M_3) < 0\right) = P(M_1 + M_2 + M_3 < 0) = P(6 + E_1 + E_2 + E_3 < 0)$$
$$= P(E_1 + E_2 + E_3 < -6)$$

this has mean $0 + 0 + 0 = 0$
$$s.d. = \sqrt{1.5^2 + 1.5^2 + 1.5^2} = 2.598$$

this probability $= \boxed{0.010}$

GO ON TO THE NEXT PAGE.

Question 3

Overview

The primary goals of this question were to assess a student's ability to: (1) recognize the random variable of interest, identify its probability distribution, and calculate a probability for a linear combination of a normal random variable and a constant; (2) use basic probability rules to find a different probability; and (3) use the sampling distribution of the sample mean to find a probability about the mean of three observations.

Sample: 3A
Score: 4

In part (a) the depth measurement, *M*, is clearly indicated as the variable of interest. The notation is clear and consistent and includes a sketch showing the normality of the distribution of the depth measurement, the *z*-score, and the shaded tail region. In part (b) the complement of the answer in part (a) is used to calculate the probability that at least one of the depth measurements will be negative. Parameters are identified, and the steps in the solution are clear. In part (c) a drawing is used to show the normality of the sampling distribution. The sketch includes the *z*-score and the shaded tail region. The work for calculating the mean and standard deviation of the sampling distribution is clear. There is a minor notation error, $P(X < 0),$ that was ignored in evaluating the overall essay. The correct notation should be $P(\overline{X} < 0).$ This essay earned a score of 4.

Sample: 3B
Score: 3

In part (a) the distribution of error, *E*, is used as the variable of interest. Since the measurement error is also normally distributed, full credit is given for this response. A drawing is used to indicate the normality of the distribution, the value of the mean, and the shaded tail region. Calculator syntax is given to compute the correct probability, but the standard deviation is not specified. In part (b) the binomial distribution is used to calculate the correct probability that at least one of three independent depth measurements will be negative. The correct calculation is shown without using the word binomial, but the values of *n* and *p* are clear from the work. In part (c) the distribution of the sum of three measurements is used to calculate the probability. The question asked the student to calculate the probability that the mean of the three independent depth measurements taken at the point where the true depth is 2 feet would be negative, but since the sum of the three measurements is also normally distributed, this approach to the solution was given full credit. The correct value for the mean and standard deviation for the sum is given. A drawing is used to indicate the distribution is normal, the value of the mean, and a shaded tail region. All parameters are defined in the calculator syntax. This essay earned a score of 3.

Sample: 3C
Score: 2

In part (a) the distribution of error, *E*, is used as the variable of interest. Since the measurement error is also normally distributed, full credit was given for this response. The essay uses correct notation, calculates the correct probability, and identifies the value of the mean and standard deviation in the substitution step of the *z*-score formula. In part (b) the binomial distribution is used to calculate the correct probability that at least one of three independent depth measurements will be negative. The essay correctly states the conditions for a binomial distribution and calculates the correct probability, but the student fails to give the parameter value *p*. In part (c)

students were asked to calculate the probability that the mean of the three independent depth measurements taken at the point where the true depth is 2 feet would be negative, but since the sum of the three measurements is also normally distributed, this student's approach to the solution was given full credit. The values for the mean and standard deviation are labeled and the correct probability is calculated, but the fact that the distribution of the sum of three independent depth measurements is normal is not shown. Although correct probability is given, it is not clear how the answer is calculated. This essay earned a score of 2.