Question 2

Intent of Question

The primary goals of this question were to assess a student's ability to (1) calculate a percentile value from a normal probability distribution; (2) recognize a binomial scenario and calculate an appropriate probability; and (3) use the sampling distribution of the sample mean to find a probability for the mean of five observations.

Solution

Part (a):

Let X denote the stopping distance of a car with new tread tires where X is normally distributed with a mean of 125 feet and a standard deviation of 6.5 feet. The z-score corresponding to a cumulative probability of 70 percent is $z = 0.52$. Thus, the 70th percentile value can be computed as:

$$x = \mu_X + z\sigma_X = 125 + 0.52(6.5) = 128.4 \text{ feet.}$$

Part (b):

From part (a), it was found that a stopping distance of 128.4 feet has a cumulative probability of 0.70. Thus the probability of a stopping distance greater than 128.4 is $1 - 0.70 = 0.30$.

Let Y denote the number of cars with the new tread pattern out of five cars that stop in a distance greater than 128.4 feet. Y is a binomial random variable with $n = 5$ and $p = 0.30$.

$$P(Y \geq 2) = 1 - P(Y \leq 1) = 1 - \left[\binom{5}{0}(0.3)^0(0.7)^5 + \binom{5}{1}(0.3)^1(0.7)^4\right]$$
$$= 1 - 0.5282 = 0.4718.$$

Part (c):

Let $\bar{X}$ denote the mean of the stopping distances of five randomly selected cars. All tires have the new tread pattern. Because the stopping distance for each of the five cars has a normal distribution, the distribution of $\bar{X}$ is normal with a mean of 125 feet and a standard deviation of $\frac{6.5}{\sqrt{5}} = 2.91$ feet. Thus,

$$P\left(\bar{X} > 130\right) = P\left(Z > \frac{130 - 125}{6.5/\sqrt{5}}\right) \approx P\left(Z > 1.72\right) = 0.0427.$$

Scoring

Parts (a), (b), and (c) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Question 2 (continued)

Part (a) is scored as follows:

Essentially correct (E) if the student clearly indicates which distribution is being used, along with the parameters (μ_X and σ_X), and correctly calculates the percentile value with appropriate justification (except for minor arithmetic or transcription errors). There are three components: distribution, parameters, and calculation of distance.

Notes:

- The standard notation $N(125,\ 6.5)$ defines distribution and parameters. Also, the z-score formula setup implies distribution and parameters. This applies only in part (a), because approximate normality is given in the stem of the problem.
- If the calculator command invNorm(0.70, 125, 6.5) is provided along with 128.4 feet *AND* an appropriately labeled sketch of a normal distribution is supplied, then the response should be scored as essentially correct (E). An appropriately labeled sketch must include correct labels for center and spread.
- If the calculator command invNorm(0.70, $\mu = 125$, $\sigma = 6.5$) is provided along with 128.4 feet, then the response should be scored as essentially correct (E).

Partially correct (P) if the student correctly supplies only two out of the three components.

Note: If the calculator command invNorm(0.70, 125, 6.5) is provided along with 128.4 feet, then the response should be scored as partially correct (P).

Incorrect (I) if the student correctly supplies at most one of the components.

Part (b) is scored as follows:

Essentially correct (E) if the student recognizes this probability as an application of the binomial distribution and sets up the problem correctly by first finding the probability for p, the probability of a success, and then using this p to find the correct binomial probability. There are three components: distribution, parameters, and calculation.

Note: If the calculator command 1-binomcdf(5, 0.3, 1) is provided along with 0.4718 and an identification of the distribution and its parameters—e.g., by the standard notation $B(5,0.3)$ or Bin(5, 0.3)—then the response should be scored as essentially correct (E).

Partially correct (P) if the student correctly supplies only two out of the three components.

Notes:

- As long as the student identifies the distribution and parameters—e.g., by the standard notation $B(5,\ 0.3)$ or Bin(5, 0.3)—the binomial formula does not need to be set up to receive full credit. However, the binomial formula setup can suffice for identifying two of the three components: distribution and parameters.
- If the calculator command 1-binomcdf(5, 0.3, 1) is provided along with 0.4718, then the response should be scored as partially correct (P).

Incorrect (I) if the student correctly supplies at most one of the components.

Question 2 (continued)

Part (c) is scored as follows:

Essentially correct (E) if the student recognizes that the distribution of the sample mean will be approximately normal with the appropriate mean and standard deviation and calculates the probability correctly. There are three components: sampling distribution, parameters, and calculation.

Notes:
- The z-score formula setup suffices only for parameters in part (c).
- If the calculator command Normalcdf(130, ∞, 125, 2.91) *AND* an appropriately labeled sketch of a normal distribution are provided along with the value obtained using the calculator, 0.0428, then the response should be scored as essentially correct (E). An appropriately labeled sketch must include correct labels for center and spread.
- If the calculator command Normalcdf(130, ∞, $\mu = 125$, $\sigma = 2.91$) is provided along with 0.0428, then the response should be scored as essentially correct (E).

Partially correct (P) if the student correctly supplies only two out of the three components.

Note: If the calculator command Normalcdf(130, ∞, 125, 2.91) is provided along with 0.0428, then the response should be scored as partially correct (P).

Incorrect (I) if the student correctly supplies at most one of the components.

Notes:
- The calculator solution is 0.0428. If this is the only information provided, the response is scored as incorrect (I).
- If a t distribution is used, then the response should be scored as incorrect (I).

A student should be penalized only once for using calculator syntax—that is, look at parts (a), (b), and (c) together.

4 Complete Response

All three parts essentially correct

3 Substantial Response

Two parts essentially correct and one part partially correct

2 Developing Response

Two parts essentially correct and no part partially correct

OR

One part essentially correct and one or two parts partially correct

OR

Three parts partially correct

Question 2 (continued)

1 **Minimal Response**

One part essentially correct and no parts partially correct

OR

No parts essentially correct and two parts partially correct

2. A tire manufacturer designed a new tread pattern for its all-weather tires. Repeated tests were conducted on cars of approximately the same weight traveling at 60 miles per hour. The tests showed that the new tread pattern enables the cars to stop completely in an average distance of 125 feet with a standard deviation of 6.5 feet and that the stopping distances are approximately normally distributed.

2A

$\mu = 125 \quad \sigma = 6.5$

(a) What is the 70th percentile of the distribution of stopping distances?

$P = .7, \quad z = .52$

$$\frac{x-\mu}{\sigma} = \frac{x-125}{6.5} = .52$$

$$x - 125 = 3.38$$

$$x = \boxed{128.38 \text{ ft}}$$

(b) What is the probability that at least 2 cars out of 5 randomly selected cars in the study will stop in a distance that is greater than the distance calculated in part (a) ?

$k=2, k=3, k=4, k=5$.

$P(x > 128.38) = .30$

Binomial probability $\quad P = .3, \quad n=5, \quad k \geq 2. \quad P(x=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$1 - (P(x=0) + P(x=1)) = 1 - \left(\binom{5}{0} \cdot .3^0 \cdot .7^5 + \binom{5}{1} \cdot .3^1 \cdot .7^4\right)$$

$$= 1 - .52822 = \boxed{.47178}$$

(c) What is the probability that a randomly selected sample of 5 cars in the study will have a mean stopping distance of at least 130 feet?

$n = 5$

$\bar{x} = 130$

$\sigma = 6.5$

$$z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{130-125}{6.5/\sqrt{5}} = 1.72$$

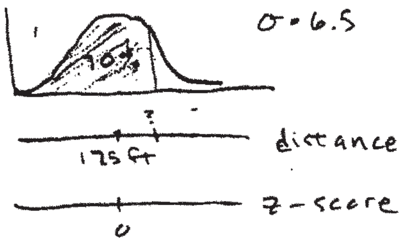
$$P(z > 1.72) = 1 - .9573 = \boxed{.0427}$$

GO ON TO THE NEXT PAGE.

2. A tire manufacturer designed a new tread pattern for its all-weather tires. Repeated tests were conducted on cars of approximately the same weight traveling at 60 miles per hour. The tests showed that the new tread pattern enables the cars to stop completely in an average distance of 125 feet with a standard deviation of 6.5 feet and that the stopping distances are approximately normally distributed.

2B

(a) What is the 70th percentile of the distribution of stopping distances?



Using invNorm(,7) on the calculator, 70% falls to the left of a z-score of .5244. So the 70th percentile is: $.5244(6.5)+125 = \boxed{128.41 \text{ ft}}$.

(b) What is the probability that at least 2 cars out of 5 randomly selected cars in the study will stop in a distance that is greater than the distance calculated in part (a) ?

Probability greater: $1-.7 = .3$.

Using binomials: Probability $= \binom{5}{2}(.3)^2(.7)^3 + \binom{5}{3}(.3)^3(.7)^2 + \binom{5}{4}(.3)^4(.7)^1 + \binom{5}{5}(.3)^5(.7)^0$

$= \boxed{.4718}$

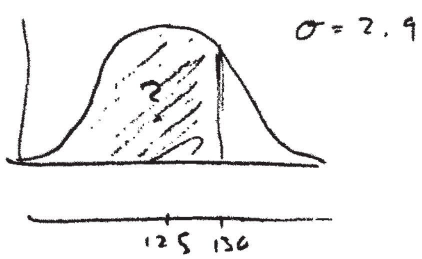
(c) What is the probability that a randomly selected sample of 5 cars in the study will have a mean stopping distance of at least 130 feet?

Looking at a sample distibution:

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{6.5}{\sqrt{5}} = 2.9$

$\bar{x} = 125$

Checking:
- randomly selected ✓
- normal population ✓



The z-score is: $\frac{130-125}{2.9} = 1.724$

Using a calculator, $\boxed{95.76\%}$ of the area is to the left of $z = 1.724$. Therefore the probability is $\boxed{95.76\%}$

GO ON TO THE NEXT PAGE.

2. A tire manufacturer designed a new tread pattern for its all-weather tires. Repeated tests were conducted on cars of approximately the same weight traveling at 60 miles per hour. The tests showed that the new tread pattern enables the cars to stop completely in an average distance of 125 feet with a standard deviation of 6.5 feet and that the stopping distances are approximately normally distributed.

2C

(a) What is the 70th percentile of the distribution of stopping distances?

$z = .52 = \frac{\bar{x} - 125}{6.5}$

70th percentile corresponds to a z score of 0.52

$\mu = 125$
$\sigma = 6.5$

$3.38 = x - 125$

$x = 128.38$ feet

The 70th percentile ~~of~~ of the distribution of stopping cars represents the stopping distances at or below 70% of the tests. This value would be approximately 128.38 feet.

(b) What is the probability that at least 2 cars out of 5 randomly selected cars in the study will stop in a distance that is greater than the distance calculated in part (a) ?

Binomial Cumulative Distribution Function

$P(X \geq 2) = \binom{5}{2}(.30)^2(1-0.30)^{5-2} + \binom{5}{2}(.30)^3(1-.30)^{5-3} + \_\_ + \_\_$

$P(X \geq 2) = 0.47178 \approx 47.18\%$

(c) What is the probability that a randomly selected sample of 5 cars in the study will have a mean stopping distance of at least 130 feet?

$n = 5$

$z = \frac{130 - 125}{6.5} = 0.77$

P(stopping at 130 ft.) = 0.7794

GO ON TO THE NEXT PAGE.

Question 2

Overview

The primary goals of this question were to assess a student's ability to (1) calculate a percentile value from a normal probability distribution; (2) recognize a binomial scenario and calculate an appropriate probability; and (3) use the sampling distribution of the sample mean to find a probability for the mean of five observations.

Sample: 2A
Score: 4

Each part of this response is complete and clearly communicated. In part (a) the student determines the z-score corresponding to a cumulative probability of 70 percent, stating that "$p = .7$" and "$z = .52$." The student then uses the z-score to correctly calculate the stopping distance. Part (a) was scored as essentially correct. In part (b) the student gives the probability of a car exceeding the stopping distance of 128.38 feet. The student indicates that the problem requires a binomial distribution. The student goes on to list the parameters and the correct values for the parameters of the binomial distribution and writes the standard formula for the binomial distribution. The student then uses the binomial formula to calculate the correct probability. Part (b) was scored as essentially correct. In part (c) the student lists the sample size, the sample statistic, and the standard deviation. The student then uses this information to construct the z-score, gives the correct probability using the calculated z-score, and correctly calculates the required probability. Part (c) was scored as essentially correct. The entire answer, based on all three parts, was judged a complete response and earned a score of 4 points.

Sample: 2B
Score: 3

Two of the three parts of this response are complete and clearly communicated. In part (a) the student sketches an approximately normal distribution, indicates the 70th percentile and the mean, and writes the standard deviation. The student uses the "invNorm" calculator command to determine the z-score of 0.5244. The student then uses the z-score formula to calculate a correct stopping distance of 128.41 feet. Part (a) was scored as essentially correct. In part (b) the student uses the binomial with the correct parameter values for selecting from 2 up to 5 cars and correctly calculates the required probability. Part (b) was scored as essentially correct. In part (c) the student states the correct standard error, gives a correct z-score, sketches the distribution, but fails to recognize that the probability should be for a mean stopping distance greater than 130 feet. The probability calculated in the response is for a mean stopping distance less than 130 feet. Part (c) was scored as partially correct. With two parts essentially correct and one part partially correct, the entire answer was judged a substantial response and earned a score of 3 points.

Sample: 2C
Score: 2

Two of the three parts of this response are complete and clearly communicated. In part (a) the student lists the parameters of the given distribution and uses the 70 percent cumulative probability to obtain the z-score of 0.52. The student uses the z-score formula to determine the correct stopping distance. The student goes on to give a complete statement indicating the required stopping distance, which shows very good communication skills. Part (a) was scored as essentially correct. In part (b) the student indicates a binomial cumulative distribution function and gives a probability statement for $X \geq 2,$ setting this

Question 2 (continued)

probability equal to the required binomial terms and summing them to get the required probability. Part (b) was scored as essentially correct. In part (c) the student fails to determine the sampling distribution and the parameters of the distribution and does not compute the correct probability. Part (c) was scored as incorrect. With two parts essentially correct and one part incorrect, the entire answer was judged a developing response and earned a score of 2 points.