Question 3

Intent of Question

The primary goals of this question were to assess a student's ability to (1) perform a probability calculation from a discrete random variable; (2) calculate the expected value of a discrete random variable; (3) perform a conditional probability calculation from a discrete random variable; and (4) use probabilistic thinking to make a prediction about how an expected value will change given a condition about the random variable.

Solution

Part (a):

The probability that at least one ATM is working when the mall opens is:

$$P(X \geq 1) = 0.21 + 0.40 + 0.24 = 0.85.$$

Part (b):

The expected value of the number of ATMs that are working when the mall opens is:

$$E(X) = 0(0.15) + 1(0.21) + 2(0.40) + 3(0.24) = 1.73 \text{ machines.}$$

Part (c):

The probability that all three ATMs are working when the mall opens, given that at least one ATM is working is:

$$P(X = 3 \mid X \geq 1) = \frac{P(X = 3 \text{ and } X \geq 1)}{P(X \geq 1)} = \frac{P(X = 3)}{P(X \geq 1)} = \frac{0.24}{0.85} \approx 0.282$$

Part (d):

Given that at least one ATM is working when the mall opens, the expected value of the number of working ATMs would be greater than the expected value calculated in part (b). By eliminating the possibility of 0 working ATMs, the probabilities for 1, 2, and 3 working ATMs all increase proportionally, so the expected value must increase.

Scoring

Parts (a), (b), (c), and (d) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Part (a) is scored as follows:

Essentially correct (E) if the probability is computed correctly with work shown.

Partially correct (P) if the correct answer is given, but no work is shown;
OR
if appropriate work is shown but the answer is incorrect or missing;
OR

Question 3 (continued)

if one of the incorrect cumulative probabilities $P(X < 1)$, $P(X \leq 1)$, or $P(X > 1)$ is stated *AND* computed correctly.

Incorrect (I) if the response does not meet the criteria for E or P.

Note: The probability can be calculated as $1 - P(X = 0) = 1 - 0.15 = 0.85$.

Part (b) is scored as follows:

Essentially correct (E) if the expected value is computed correctly with work shown.

Partially correct (P) if the correct answer is given, but no work is shown;
OR
if appropriate work is shown but the answer is incorrect or missing.

Incorrect (I) if the response does not meet the criteria for E or P.

Part (c) is scored as follows:

Essentially correct (E) if the probability is computed correctly, with work shown that includes correct numerical values for both the numerator and denominator.

Partially correct (P) if the response includes a numerator and denominator in calculating the conditional probability, with one (numerator or denominator) correct in numerical value and the other incorrect;
OR
if the correct answer is given, but no work is shown.

Incorrect (I) if the response does not meet the criteria for E or P.

Part (d) is scored as follows:

Essentially correct (E) if the response provides the correct answer (greater than) with a reasonable explanation based on the fact that with $X = 0$ eliminated, the probabilities for $X = 1$, $X = 2$, and $X = 3$ *all* increase;
OR
if the response provides the correct answer (greater than) with a reasonable explanation based on the fact that with $X = 0$ eliminated, the balance point of the distribution increases;
OR
if the response provides the correct answer (greater than) and the conditional expected value is computed correctly with work shown.

Partially correct (P) if the response provides the correct answer (greater than) with a weak explanation, such as "Yes, because 0 is eliminated";
OR

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if the response provides the correct answer (greater than) with a reasonable but incorrect attempt to calculate the conditional expected value using a revised probability distribution with only the values $X = 1$, $X = 2$, and $X = 3$, and corresponding probabilities greater than or equal to those in the original probability distribution.

Incorrect (I) if the response does not provide the correct answer (greater than);
OR
if the response provides the correct answer (greater than) with an incorrect explanation or no explanation.

Note: The conditional expected value is:

$$1\left(\frac{0.21}{0.85}\right) + 2\left(\frac{0.40}{0.85}\right) + 3\left(\frac{0.24}{0.85}\right) \approx 1(0.247) + 2(0.471) + 3(0.282) = 2.04 \text{ machines.}$$

Each essentially correct (E) part counts as 1 point. Each partially correct (P) part counts as ½ point.

4 Complete Response

3 Substantial Response

2 Developing Response

1 Minimal Response

If a response is between two scores (for example, 2½ points), use a holistic approach to decide whether to score up or down, depending on the overall strength of the response and communication.

A 3A1

3. A shopping mall has three automated teller machines (ATMs). Because the machines receive heavy use, they sometimes stop working and need to be repaired. Let the random variable X represent the number of ATMs that are working when the mall opens on a randomly selected day. The table shows the probability distribution of X.

Number of ATMs working when the mall opens	0	1	2	3
Probability	0.15	0.21	0.40	0.24

(a) What is the probability that at least one ATM is working when the mall opens?

P(at least one) = 1 − P(no ATMs) = 1 − .15 = 0.85

(b) What is the expected value of the number of ATMs that are working when the mall opens?

E(x) = (0)(.15) + (1)(.21) + (2)(.4) + (3)(.24) = 1.730 ATMs

(c) What is the probability that all three ATMs are working when the mall opens, given that at least one ATM is working?

$P(\text{all 3 work} \mid \text{at least one works}) = \frac{P(\text{all 3 work} \cap \text{at least one works})}{P(\text{at least one works})} = \frac{.24}{.85} = .282$

(d) Given that at least one ATM is working when the mall opens, would the expected value of the number of ATMs that are working be less than, equal to, or greater than the expected value from part (b) ? Explain.

The expected value for number of working ATMs would be greater than that of part B. Because at least one ATM must be working, zero is no longer an x-value in the probability distribution. Thus, in the distribution, the probabilities of having 1, 2, and 3 ATMs all go up. When finding the E(x) by doing E(x) = ∑ (x)·p(x), because all the p(x) values increase, the sum increases and E(x) is greater.

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B 3B1

3. A shopping mall has three automated teller machines (ATMs). Because the machines receive heavy use, they sometimes stop working and need to be repaired. Let the random variable X represent the number of ATMs that are working when the mall opens on a randomly selected day. The table shows the probability distribution of X.

Number of ATMs working when the mall opens	0	1	2	3
Probability	0.15	0.21	0.40	0.24

(a) What is the probability that at least one ATM is working when the mall opens?

$.21 + .40 + .24 = \boxed{.85}$

(b) What is the expected value of the number of ATMs that are working when the mall opens?

$E(x) = (0 \cdot .15) + (1 \cdot .21) + (2 \cdot .40) + (3 \cdot .24)$

$\boxed{E(x) = 1.73 \text{ atms}}$

(c) What is the probability that all three ATMs are working when the mall opens, given that at least one ATM is working?

$\frac{.24}{.85} = \boxed{.282}$

(d) Given that at least one ATM is working when the mall opens, would the expected value of the number of ATMs that are working be less than, equal to, or greater than the expected value from part (b) ? Explain.

Equal to because Expected value is calculated by Σ(number of ATM's) × (probability) so (0 × .15) = 0.

C

3C

3. A shopping mall has three automated teller machines (ATMs). Because the machines receive heavy use, they sometimes stop working and need to be repaired. Let the random variable X represent the number of ATMs that are working when the mall opens on a randomly selected day. The table shows the probability distribution of X.

Number of ATMs working when the mall opens	0	1	2	3
Probability	0.15	0.21	0.40	0.24

(a) What is the probability that at least one ATM is working when the mall opens?

0.85

(b) What is the expected value of the number of ATMs that are working when the mall opens?

EV = 0(.15) + 1(.21) + 2(.40) + 3(.24)

EV = 1.73

(c) What is the probability that all three ATMs are working when the mall opens, given that at least one ATM is working?

P(A|B) = 0.24 / 0.85 = 0.289

(d) Given that at least one ATM is working when the mall opens, would the expected value of the number of ATMs that are working be less than, equal to, or greater than the expected value from part (b) ? Explain.

The expected value would be equal to that in part b, because the probability of 0 ATMs working when the mall opens does not affect the Expected Value of part b because you multiply by zero, and it also is not included in part d, so the expected values would remain the same.

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Question 3

Overview

The primary goals of this question were to assess a student's ability to (1) perform a probability calculation from a discrete random variable; (2) calculate the expected value of a discrete random variable; (3) perform a conditional probability calculation from a discrete random variable; and (4) use probabilistic thinking to make a prediction about how an expected value will change given a condition about the random variable.

Sample: 3A
Score: 4

In part (a) the response calculates the correct probability that at least one ATM is working when the mall opens with appropriate supporting work based on the complement rule. Part (a) was scored as essentially correct. Part (b) includes the correct expected value of 1.73 ATMs with appropriate work shown. Part (b) was scored as essentially correct. The response computes the conditional probability correctly, with the proper numerator and denominator shown, and part (c) was scored as essentially correct. In part (d) the response gives a clear explanation of why the expected value would increase based on the fact that the probabilities for $X = 1$, $X = 2$, and $X = 3$ would all increase. Consequently, part (c) was scored as essentially correct. Because all four parts were scored as essentially correct, the response earned a score of 4.

Sample: 3B
Score: 3

In part (a) the response correctly computes $P(X \geq 1)$ by adding the probabilities for $X = 1$, $X = 2$, and $X = 3$. With a correct answer of 0.85 and work shown, part (a) was scored as essentially correct. Part (b) includes a proper calculation of the expected value that results in the correct answer of 1.73 ATMs. Part (b) was scored as essentially correct. In part (c) the response computes the conditional probability correctly, with the proper numerator and denominator shown. Part (c) was scored as essentially correct. In part (d) the response incorrectly concludes that the expected value would be equal to the value from part (b). Consequently, part (d) was scored as incorrect. Because three parts were scored as essentially correct, and one part was scored as incorrect, the response earned a score of 3.

Sample: 3C
Score: 2

In part (a) the response provides the correct answer with no supporting work. As a result, part (a) was scored as partially correct. In part (b) the expected value is computed correctly with appropriate work shown. Part (b) was scored as essentially correct. In part (c) the response gives the correct numerator and denominator in the conditional probability formula but gets an incorrect answer of 0.289 instead of 0.282. Due to the minor arithmetic error, part (c) was scored as partially correct. In part (d) the response incorrectly states that the expected value would be equal to the value from part (b), so part (d) was scored as incorrect. Because one part was scored as essentially correct, two parts were scored as partially correct, and one part was scored as incorrect, the response earned a score of 2.