2018

AP CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

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Question 5

(a) The average rate of change of f on the interval $0 \le x \le \pi$ is $\dfrac{f(\pi) - f(0)}{\pi - 0} = \dfrac{-e^{\pi} - 1}{\pi}$.	1 : answer
(b) $f'(x) = e^x \cos x - e^x \sin x$ $f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2} \cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2} \sin\left(\frac{3\pi}{2}\right) = e^{3\pi/2}$ The slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$ is $e^{3\pi/2}$.	$2 : \begin{cases} 1 : f'(x) \\ 1 : \text{slope} \end{cases}$
(c) $f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow x = \frac{\pi}{4}, x = \frac{5\pi}{4}$	$3 : \begin{cases} 1 : \text{sets } f'(x) = 0 \\ 1 : \text{identifies } x = \frac{\pi}{4}, x = \frac{5\pi}{4} \\ \quad \text{as candidates} \\ 1 : \text{answer with justification} \end{cases}$

x	$f(x)$
0	1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}} e^{\pi/4}$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}} e^{5\pi/4}$
2π	$e^{2\pi}$

The absolute minimum value of f on $0 \le x \le 2\pi$ is $-\frac{1}{\sqrt{2}} e^{5\pi/4}$.

(d) $\lim\limits_{x \to \pi/2} f(x) = 0$ Because g is differentiable, g is continuous. $\lim\limits_{x \to \pi/2} g(x) = g\left(\frac{\pi}{2}\right) = 0$ By L'Hospital's Rule, $\lim\limits_{x \to \pi/2} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to \pi/2} \dfrac{f'(x)}{g'(x)} = \dfrac{-e^{\pi/2}}{2}$.	$3 : \begin{cases} 1 : g \text{ is continuous at } x = \frac{\pi}{2} \\ \quad \text{and limits equal } 0 \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$ Note: max $1/3$ [1-0-0] if no limit notation attached to a ratio of derivatives

5A
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5. Let f be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of f on the interval $0 \le x \le \pi$.

$$\frac{\int_0^{\pi} f'(x)\,dx}{\pi} = \frac{f(\pi) - f(0)}{\pi} = \frac{e^{\pi}\cos\pi - e^0\cos 0}{\pi}$$

$$= \frac{e^{\pi}(-1) - 1(1)}{\pi} = \boxed{\frac{-e^{\pi} - 1}{\pi}}$$

(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

$$\frac{df}{dx} = e^x \cos x + e^x(-\sin x)$$

$$= e^x(\cos x - \sin x)$$

$$e^{3\pi/2}(0 - (-1)) = \boxed{e^{3\pi/2}}$$

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(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

$f'(x) = e^x(\cos x - \sin x) = 0 \rightarrow \cancel{e^x = 0}$ or $\cos x = \sin x$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$.

From part B

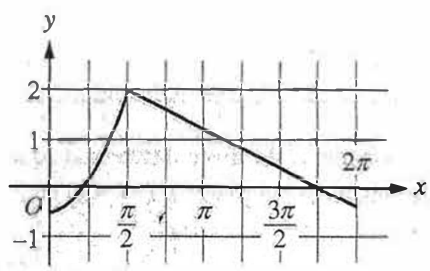
Minima either have a derivative of 0, undefined, or are bordering points.

So, possible values: $x = 0, x = 2\pi, x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

$f(x) = 1$ $f(x) = e^{2\pi}$ $f(x) = e^{\pi/4}\left(\frac{\sqrt{2}}{2}\right)$ $f(x) = e^{5\pi/4}\left(-\frac{\sqrt{2}}{2}\right)$.

Since $e^{5\pi/4}\left(-\frac{\sqrt{2}}{2}\right)$ is the only negative value of $f(x)$, the absolute minimum of $f(x)$ on $0 \le x \le 2\pi$ is $\boxed{-\frac{\sqrt{2}}{2}e^{5\pi/4}}$.

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim\limits_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

Note $\lim\limits_{x \to \frac{\pi}{2}} (g(x)) = 0$ since $g(x)$ is differentiable (and thus continuous), and $g(\frac{\pi}{2}) = 0$. Also $f(\frac{\pi}{2}) = e^{\pi/2}(\cos(\frac{\pi}{2})) = e^{\pi/2}(0) = 0$, and $f(x)$ is continuous, so $\lim\limits_{x \to \frac{\pi}{2}} f(x) = 0$. So, by L'Hopital's rule,

$$\lim_{x \to \frac{\pi}{2}} \left(\frac{f(x)}{g(x)}\right) = \lim_{x \to \frac{\pi}{2}} \left(\frac{f'(x)}{g'(x)}\right) = \frac{e^{\pi/2}\left(\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right)}{2} = \boxed{\frac{-e^{\pi/2}}{2}}$$

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NO CALCULATOR ALLOWED

5B

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5. Let f be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of f on the interval $0 \leq x \leq \pi$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{-e^{\pi} - 1}{\pi} = \boxed{-\frac{e^{\pi}}{\pi} - \frac{1}{\pi}}$$

(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

$$f'(x) = -e^x \sin x + e^x \cos x$$

$$f'\left(\frac{3\pi}{2}\right) = -e^{3\pi/2}(-1) + e^{3\pi/2}(0)$$

$$= \boxed{e^{3\pi/2}}$$

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NO CALCULATOR ALLOWED

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(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

$f'(x) = -e^x \sin x + e^x \cos x = 0$

$\cancel{-e^x} \sin x = \cancel{-e^x} \cos x$

$x = \pm \frac{\pi}{4}$

$f(0) = 1$

$f(2\pi) = e^{2\pi}$

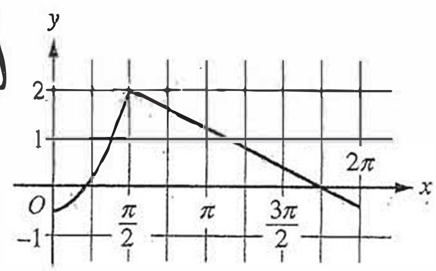
$f(-\frac{\pi}{4}) = e^{-\pi/4}(-\frac{\sqrt{2}}{2})$

$f(\frac{\pi}{4}) = e^{\pi/4}(\frac{\sqrt{2}}{2})$

The absolute minimum value is $-\frac{\sqrt{2}e^{-\pi/4}}{2}$ because $\pm\frac{\pi}{4}$ are critical values and $f(-\pi/4) > f(0)$, $f(2\pi)$, and $f(\pi/4)$

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim\limits_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.

$\lim\limits_{x \to \pi/2} \frac{f'(x)}{g'(x)} = \boxed{-\frac{e^{\pi/2}}{2}}$



Graph of g'

$g'(\pi/2) = 2$

$f'(\pi/2) = -e^{\pi/2}(1) + e^{\pi/2}\cos$
$= -e^{\pi/2}$

5C
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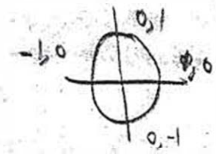
5. Let f be the function defined by $f(x) = e^x \cos x$.



(a) Find the average rate of change of f on the interval $0 \le x \le \pi$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{e^{\pi}\cos(\pi) - e^{0}\cos(0)}{\pi}$$

$$= \frac{-e^{\pi} - 1}{\pi}$$

(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

$f(\frac{3\pi}{2}) = e^{\frac{3\pi}{2}} \cos(\frac{3\pi}{2})$

$= 0$

$f'(x) = e^{x}\cos(x) - e^{x}\sin(x)$

$f'(\frac{3\pi}{2}) = e^{\frac{3\pi}{2}}\cos(\frac{3\pi}{2}) - e^{\frac{3\pi}{2}}\sin(\frac{3\pi}{2})$

$e^{\frac{3\pi}{2}}(0) - e^{\frac{3\pi}{2}}(1)$

$= -e^{\frac{3\pi}{2}}$

$y = -e^{\frac{3\pi}{2}}(x - \frac{3\pi}{2})$

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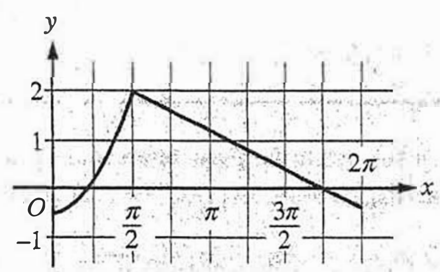
(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

$f'(x) = e^x \cos(x) - e^x \sin(x) = 0$

$e^x \left(\cos(x) - \sin(x)\right) = 0$

the absolute min value is when $f'(x) = 0$ and changes signs from $-$ to $+$

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

$\lim_{x \to \frac{\pi}{2}} f(x) = f'\left(\frac{\pi}{2}\right) = -e^{\frac{\pi}{2}}$

$\lim_{x \to \frac{\pi}{2}} g(x) = 2$

$g\left(\frac{\pi}{2}\right) = 0$

limit exists b/c the

$\lim_{x \to \frac{\pi}{2}} \frac{f(x)}{g(x)} = \lim_{x \to \frac{\pi}{2}} \frac{-e^{\frac{\pi}{2}}}{2}$

Question 5

Overview

In this problem the function f is defined by $f(x) = e^x \cos x$. In part (a) students were asked for the average rate of change of f on the interval $0 \le x \le \pi$. A correct response should demonstrate that the average rate of change is found using a difference quotient, $\frac{f(\pi) - f(0)}{\pi - 0}$. In part (b) students were asked for the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$. A correct response should use the fact that the slope of the tangent line is the value of the derivative of f at the indicated point. The given expression for $f(x)$ can be differentiated using the product rule. In part (c) students were asked to find, with justification, the absolute minimum value of f on the interval $0 \le x \le 2\pi$. The Extreme Value Theorem guarantees that f attains a minimum on the interval $0 \le x \le 2\pi$. Candidates for locations of this minimum value are the endpoints of the interval and critical points for f inside the interval. In this case, f is differentiable, so critical points are the two zeros of f', $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. The absolute minimum value of f is the least of the values of f at the four candidates. In part (d) it is given that g is a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$, and the graph of the derivative of g for $0 \le x \le 2\pi$ is supplied. Students were asked to find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$, if it exists, and to justify their answer. A correct response should start with a confirmation that the requested limit is an indeterminate form to which L'Hospital's Rule applies. The numerator, $f(x)$, has limit 0 as $x \to \frac{\pi}{2}$, using limit properties and continuity of the exponential and cosine functions. Because g is differentiable, it is continuous, so the limit of the denominator, $g(x)$, can be computed by substitution, yielding $\lim_{x \to \pi/2} g(x) = 0$. After the indeterminate form is confirmed, applying L'Hospital's Rule leads to the limit $\lim_{x \to \pi/2} \frac{f'(x)}{g'(x)}$, which can be evaluated using properties of limits and the provided graph of g'.

For part (a) see LO 2.1A/EK 2.1A1. For part (b) see LO 2.1C/EK 2.1C3, LO 2.3B/EK 2.3B1. For part (c) see LO 1.2B/EK 1.2B1, LO 2.2A/EK 2.2A1. For part (d) see LO 1.1C/EK 1.1C3, LO 2.2B/EK 2.2B2. This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 5A
Score: 9

The response earned all 9 points: 1 point in part (a), 2 points in part (b), 3 points in part (c), and 3 points in part (d). In part (a) the response presents a difference quotient and would have earned the point for $\frac{e^{\pi} \cos \pi - e^0 \cos 0}{\pi}$ in line 1 with no simplification. In this case, correct simplification to $\frac{-e^{\pi} - 1}{\pi}$ earned the point. In part (b) the response would have earned the first point for $\frac{df}{dx} = e^x \cos x + e^x(-\sin x)$ in line 1. Responses that go on to factor must do so correctly. The response earned the point with the derivative expression factored correctly to $e^x(\cos x - \sin x)$. The response would have earned the second point for the slope

Question 5 (continued)

$e^{3\pi/2}(0-(-1))$ in line 3 with no simplification. In this case, correct simplification to $e^{3\pi/2}$ earned the point. In part (c) the response earned the first point with $f'(x) = e^x(\cos x - \sin x) = 0$ in line 1. The response earned the second point for identifying $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ as candidates in line 2 on the right. The response earned the third point for the absolute minimum value of $-\frac{\sqrt{2}}{2}e^{5\pi/4}$ and the global argument appealing to the behavior of $f(x)$ on the entire interval. The candidates test is used to justify that the minimum value is the only negative extreme value on the interval. In part (d) the response earned the first point by stating that $g(x)$ is continuous in line 1 and showing in lines 1 and 3 that $\lim\limits_{x\to\pi/2}(g(x)) = 0$ and $\lim\limits_{x\to\pi/2} f(x) = 0.$ The response earned the second point for a limit attached to a ratio of derivatives $\lim\limits_{x\to\pi/2}\left(\frac{f'(x)}{g'(x)}\right)$ in the last line. The third point was earned for the answer $\frac{-e^{\pi/2}}{2}$.

Sample: 5B
Score: 6

The response earned 6 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the response presents a difference quotient and would have earned the point for $\frac{-e^{\pi}-1}{\pi}$ in line 1. In this case, the rewriting of the expression as $-\frac{e^{\pi}}{\pi} - \frac{1}{\pi}$ earned the point. In part (b) the response earned the first point for $f'(x) = -e^x \sin x + e^x \cos x.$ The response would have earned the second point for the slope $f'\left(\frac{3\pi}{2}\right) = -e^{3\pi/2}(-1) + e^{3\pi/2}(0)$ in line 2 with no simplification. In this case, correct simplification to $e^{3\pi/2}$ earned the point. In part (c) the response earned the first point with $f'(x) = -e^x \sin x + e^x \cos x = 0.$ Because the response does not identify both $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ as candidates, the second point was not earned. The third point was not earned because $-\frac{\sqrt{2}e^{-\pi/4}}{2}$ is not the absolute minimum value of f on $0 \le x \le 2\pi.$ Additionally, $x = -\frac{\pi}{4}$ is not in the interval. In part (d) the response did not earn the first point because the conditions that g is continuous and that $\lim\limits_{x\to\pi/2} f(x) = \lim\limits_{x\to\pi/2} g(x) = 0$ are not verified. The response earned the second point in line 1 for a limit attached to a ratio of derivatives $\lim\limits_{x\to\pi/2}\frac{f'(x)}{g'(x)}.$ The third point was earned for the answer $-\frac{e^{\pi/2}}{2}.$

Sample: 5C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the response presents a difference quotient and would have earned the point for $\frac{e^{\pi}\cos(\pi) - e^{0}\cos(0)}{\pi}$

Question 5 (continued)

in line 1 with no simplification. In this case, correct simplification to $\frac{-e^{\pi}-1}{\pi}$ earned the point. In part (b) the response earned the first point for $f'(x) = e^x \cos(x) - e^x \sin(x)$ in line 1 on the right. The response would have earned the second point for the slope $f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2} \cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2} \sin\left(\frac{3\pi}{2}\right)$ in line 2 on the right with no simplification. In this case, incorrect simplification to $-e^{3\pi/2}$ did not earn the point. In part (c) the response earned the first point with $f'(x) = e^x \cos(x) - e^x \sin(x) = 0.$ Because the response does not identify both $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ as candidates, the second point was not earned. The third point was not earned because an absolute minimum value with a global argument is not presented. In part (d) the response did not earn the first point because the conditions that g is continuous and that $\lim\limits_{x\to\pi/2} f(x) = \lim\limits_{x\to\pi/2} g(x) = 0$ are not verified. The second point was not earned because there is no limit attached to a ratio of derivatives. A maximum of 1 point can be earned in part (d) for responses with no limit notation attached to a ratio of derivatives. As a result, the response is not eligible for the third point.